Basic Solutions of Diffusion Equation

TAKAHASHI Kunio

(with help of typing by Mr.Nakayama)

1. One dimensional diffusion equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(1a)}$$

where a is constant $u(t,x)$ is a function of time t and position x. The initial condition is

$$u(0,x) = u_0(x) \qquad \text{(1b).}$$

From the Fourier transfer of Eq.(1) with respect to the position x, we obtain

$$\frac{\partial U}{\partial t} = -a^2 \xi^2 U(t,\xi) \qquad \text{(2)}$$

where $U(t,\xi)$ is the Fourier transfer of $u(t,x)$ as

$$F(\xi) = \Im\{f\}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\exp(-i\xi x)dx \qquad \text{(3a)}$$

$$f(\xi) = \Im^{-1}\{F\}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(x)\exp(i\xi x)d\xi \qquad \text{(3b).}$$

Eq.(2) can be obtained easily using a theorem

$$F\{f^{(p)}\}(\xi) = (i\xi)^p F\{f\}(\xi) \qquad \text{(4).}$$

The solution of Eq.(2) is

$$U(t,\xi) = U_0(\xi) e^{-a^2\xi^2 t} \qquad \text{(5).}$$

From the inverse transfer of Eq.(5), we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_0(\xi) e^{-a^2\xi^2 t} e^{i\xi x} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(\alpha) e^{-i\xi\alpha} d\alpha \right) e^{-a^2\xi^2 t} e^{i\xi x} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(\alpha) e^{i\xi(x-\alpha)} e^{-a^2\xi^2 t} d\xi d\alpha \qquad \text{(6).} \end{aligned}$$

By a parameter change as $a^2\xi^2 t = \frac{\eta^2}{2}$, $d\xi = \frac{1}{\sqrt{2a^2 t}} d\eta$ and

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_0(\alpha)e^{-\frac{\eta^2}{2}}e^{i(x-\alpha)\frac{\eta}{\sqrt{2a^2t}}}\frac{1}{\sqrt{2a^2t}}d\eta d\alpha$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_0(\alpha)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{\eta^2}{2}}e^{i\eta\frac{x-\alpha}{\sqrt{2a^2t}}}d\eta\frac{1}{\sqrt{2a^2t}}d\alpha \quad (7)$$

can be used. Using another theorem

$$\Im\left\{e^{-\frac{x^2}{2}}\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{-\frac{x^2}{2}}\right)e^{-i\xi x}dx=e^{-\frac{\xi^2}{2}} \quad (8),$$

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_0(\alpha)e^{-(\frac{x-\alpha}{\sqrt{2a^2t}})^2/2}\frac{1}{\sqrt{2a^2t}}d\alpha$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_0(\alpha)\frac{1}{\sqrt{2a^2t}}e^{-\frac{(x-\alpha)^2}{4a^2t}}d\alpha$$

$$=\int_{-\infty}^{\infty}u_0(\alpha)\frac{1}{\sqrt{4\pi a^2t}}e^{-\frac{(x-\alpha)^2}{4a^2t}}d\alpha \quad (9)$$

Now, the solution of the diffusion equation under a initial condition can be obtained as a expression of an integral.

Please note That the constant a^2 is called diffusion coefficient. So, an expression as

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}$$

is used frequently. Here a^2 is used because the coefficient is limited to be positive.